# A note on the singularity in the evolution of nonlocal constraint flows

By **Xiaoliu Wang**

**(School of Mathematics, Southeast University)**

**(Email:xlwang@seu.edu.cn)**

We look at the case of Gage's area-preserving flow first. Here the initial curve is locally convex, immersed and may have intersection with itself.

(1) If the algebraic area of initial curve is negative (in the sense of usual orientation), i.e., $A_0 < 0$, it is shown that the flow produces a singularity at a finite time in the paper [1] by Sesum,Tsai and Wang.

(2) If the algebraic area of initial curve is zero, i.e., $A_0 = 0$, it is shown that the flow produces a singularity at the maximal existence time $T_{\max}$ in the same paper. Note that in this case, we do not know whether $T_{\max} < \infty$ or not? This problem is proposed in [2]. Nevertheless, if one can show that the flow does not converge to a point as $t \to T_{\max}$, then $T_{\max}$ must be finite.

(3) How about the case of $A_0 > 0$? In fact, it is proved in [1] that if the initial curve has some rotationally symmetry (and some additional assumptions if necessary), then the flow exists globally and converges to an m-fold circle at $t \to \infty$. But it is still possible for the flow to produce a singularity at a finite time, that is, when initial curve satisfies $L_0^2 < 4m\pi A_0$ (with total curvature being $2m\pi$), see the proof in [1]. We note that a limacon $r(\theta) = 1 + 1.7\sin\theta$ ($\theta \in [0, 4\pi]$) satisfies $L_0^2 < 8\pi A_0$ (see [3]) and thus the flow starting from it produces a singularity at finite time. This in fact answers an open problem proposed by Yazaki in [4].

Now, we look at the length-preserving flow proposed by Ma-Zhu [5]. In [1], the authors have shown that different from the case of area-preserving flow, it is the positivity of an energy at initial time (denoted by $E(0)$) that impacts the global existence of flow.

(1) If $E(0) < 0$ or $L_0^2 < 4m\pi A_0$, the flow produces a singularity at a finite time.

(2) If $E(0) \geq 0$ and initial curve is rotationally symmetric (without additional condition), then the flow exists globally and converges to an m-fold circle as $t \to \infty$.Recently, Bai, Li and Wang show in preprint [6] that if initial curve satisfies $E(0) \geq 0$ and is axis-symmetric, then the flow still has a similar global convergence.

**References**

[1] N. Sesum, D.H. Tsai and X.L. Wang, *Evolution of Locally Convex Closed Curves in Nonlocal Curvature Flows*, **arXiv:1708.04827v1**, 2017.

[2] J. Escher and K. Ito, *Some dynamic properties of volume preserving curvature*

*driven flows,* **Math. Ann.**, 333 (2005) 213–230.

[3] J. Escher, U.F. Mayer and G. Simonett, *The surface diffusion flow for immersed hypersurfaces*, **SIAM J. Math. Anal.**, 29 (1998) 1419-1433.

[4] S. Yazaki, *Introduction to an area-preserving crystalline curvature flow equation*, **Lecture note as Czech Technical University in Prague**, 2006.

[5] L. Ma and A.Q. Zhu, *On a length preserving curve flow*, **Monatshefte Für Math.**, 165 (2012) 57-78.

[6] X.L. Bai, F. Li and X.L. Wang, *Longtime behaviour of solutions to a parabolic type equation arising from the curvature flow*, **preprint**, 2019.